DIGITAL SIGNAL PROCESSING

UNIT-IV
IIR Filter Design

1. Concept of Filter and its Characteristics
2. Classification of Filters
   - LPF/ HPF / BPF / BSF
   - Analog / Digital Filters
   - FIR / IIR Filters
3. Aalog Filter Approximation Methods
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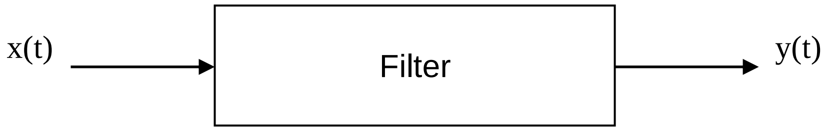
Concept of Filter and Its Characteristics:

In general, filters are used to extract required information and attenuate all other unwanted things. In communication, filter can be defined as a frequency selective device for various frequencies of signals. Characteristics of filter are

Due to filtering

- Some frequency components may be boosted in strength.
- Some frequency components may be attenuated.
- Some frequency components may be unchanged.

For a distortion less filter, the output wave shape is the exact replica of the input wave shape over specified band of frequency signals and attenuates all other unwanted frequency signals. Let us consider a filter having input x(t) and output y(t) as shown.



For a distortion less filter, the relation between input x(t) and output y(t) is $y(t) = A\ x(t - t_0)$, where A is gain and t_0 is delay.

Apply Fourier Transform both side, to obtain the frequency response

$$FT\ [\ y(t)\ ] = FT\ [\ A\ x(t - t_0)\ ]$$

$$Y(j\ \omega) = A\ FT\ [\ x(t - t_0)\ ]$$

$$Y(j\ \omega) = A\ e^{-j\ \omega t\ o}\ X(j\ \omega)$$

$$H(j\ \omega) = A\ e^{-j\ \omega t\ o}$$

It is very clear from above frequency response, for a distortion less filter the magnitude response is constant over required band of frequency signals and phase is linear.

Classification of Filters:

(A)LPF/HPF/BPF/BSF

Based on frequency response, filters are classified into four types.

- Low Pass Filters (LPF)
- High Pass Filters (HPF)
- Band Pass Filters (BPF)
- Band Stop Filters (BSF)

Low pass filter allows only low frequency signals and attenuate all other high frequency signals.

High pass filter allows only high frequency signals and attenuate all other low frequency signals.

Band pass filter allows only a certain band of frequency signals and attenuate all other unwanted band of frequency signals.

Band stop filter or Band reject filter or Band elimination filter allows entire band of frequency signals and attenuate a narrow band or unwanted band of frequency signals.

(B)Analog and Digital Filters

Based on type of input, type of output and components used, filters are classified into two types.

- Analog Filters
- Digital Filters

Analog Filters:

- An analog filter produces analog signals with an input of analog signal.

- Analog filters are described by differential equation, which involves only differentials.

Ex:

(a) A simple RC high pass filter acts as a differentiator, where the output y(t) is the differentiation of input x(t)

$$y(t) = \frac{d}{dt} [\ x(t)\ ]$$

(b) A simple RC low pass filter acts as an integrator, where the output y(t) is the integration of input x(t)

$$y(t) = \int x(t) \Rightarrow x(t) = \frac{d}{dt} [y(t)]$$

- Analog filters are constructed by using analog components, like resistors, capacitors, inductors, diodes, transistors, linear ICs, etc,.
- Thermal stability of analog filter is poor because of all analog components are temperature sensitive.
- Analog filters are non programmable and static in nature.

Digital Filters:

- Digital filter produces digital signals with an input of digital signal.

- Digital filters are described by a difference equation, which does not involve differentials, which involve only shifts.

Ex:

$$y(n) = 2\,x(n) + 3\,x(n-1) + 4\,x(n-2) \text{ and}$$
$$y(n) = 2\,x(n) + 3\,x(n-1) + 4\,y(n-2).$$

- Digital filters are constructed by using discrete components like adders, constant multipliers and delays (memories).
- Thermal stability of digital filter is high.
- Digital filters are programmable and dynamic.

(C)FIR and IIR Filters

Based on impulse response, filters are classified into two types

- Finite Impulse Response Filters (FIR) Filters
- Infinite Impulse Response Filters (IIR) Filters

The filter designed by considering only finite samples of impulse response is called Finite Impulse Response (FIR) Filter.

The filter designed by considering all the infinite samples of impulse response is called Infinite Impulse Response (IIR) Filter.

Ex 1: A discrete system having LCCDE $y(n) = 2\,x(n) + 3\,x(n-1) + 4\,x(n-2)$ is FIR system because it has finite samples of impulse response, $h(n) = \{2, 3, 4\}$.

Ex 2: A discrete system having LCCDE $y(n) = 2\,x(n) + 3\,y(n-1)$ is IIR system because it has infinite samples of impulse response, $h(n) = \{2, 6, 18, 56, \ldots\ldots\ldots\}$.

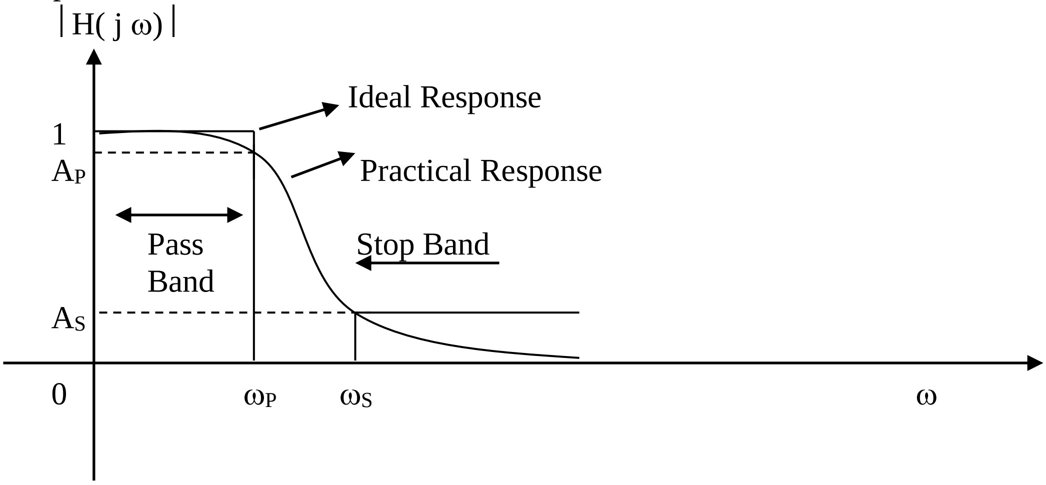
Ex 3: A discrete system having LCCDE $y(n) = 2\,x(n) + 1/3\,y(n-1$ is IIR system because it has infinite samples of impulse response, $h(n) = \{2, 2/3, 2/9, 2/27, \ldots\ldots\ldots\}$.

Analog Filter Approximation Methods:

There are many approximation procedures, which are used in the design of an analog filter from given specifications of low pass filter. Among them, popular approximation methods are

- Butterworth Approximation
- Chebyshev Approximation

Magnitude Response of LPF:



Where,

ω_P : Pass band digital frequency in rad/sample.

ω_S : Stop band digital frequency in rad/sample.

A_P : Gain at pass band digital frequency ω_P or

A_S : Gain at stop band digital frequency ω_S.

Ω_P: Pass band analog frequency in rad/sec.

Ω_S: Stop band analog frequency in rad/sec.

(A)Butterworth Approximation:

Analog Butterworth Low Pass Flter (ABLPF) is designed by approximating the magnitude of frequency response $|H(i\Omega)|$ is selected such that the magnitude is maximally flat in the pass band and monotonically decreasing in the stop band. The approximated squared magnitude frequency response of low pass filter as given by

$$|H(j\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_C}\right)^{2N}}$$

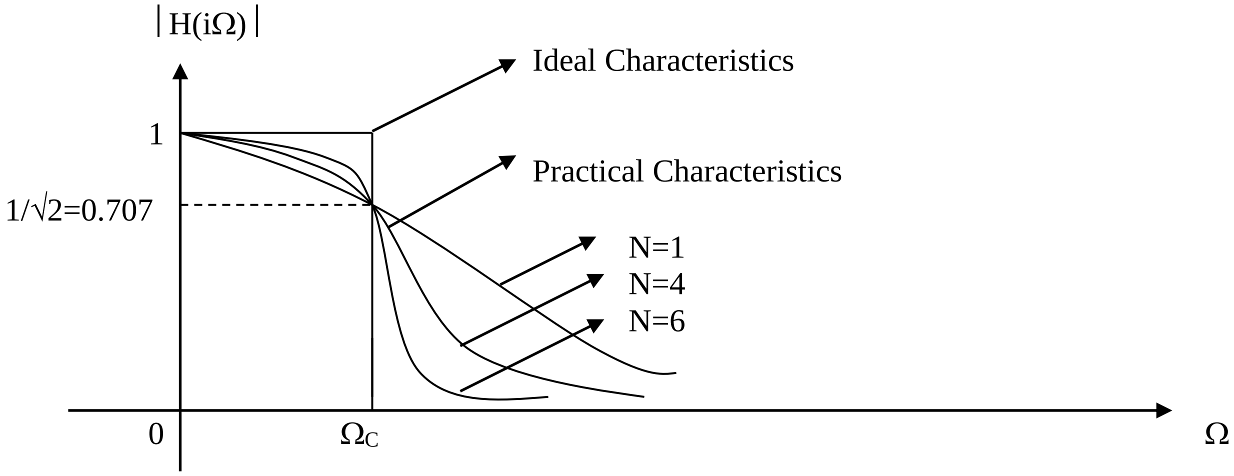
Where,

Ω_C : Cutoff frequency of an analog filter in rad/sec.

N : Order of the filter (Number of poles)

(i)Magnitude Response of ABLPF:

Magnitude response $|H(i\Omega)|$ Vs Ω for different N values as shown, where the magnitude is maximally flat in the pass band and monotonically decrease stop band when the N value increases.



(ii)Properties of ABLPF:

- Butterworth filter is a all pole design.
- All the poles lie on a circle in s-plane.
- At the cutoff frequency ($\Omega = \Omega_C$), the magnitude response $|H(i\Omega)| = 0.707$.
- Order (N) of the filter is depends on specifications of the desired filter.
- Magnitude of frequency response is maximally flat over pass band and monotonically decreases with increase in N.
- The magnitude response approaches the ideal response as the order of the filter increases.

(iii)Order of ABLPF:

Order of the filter is nothing but number of poles of desired filter and it can be computed from the formula

$$N = \frac{1}{2} \frac{\log\left[\left(\frac{1}{A_S^2} - 1\right) \Big/ \left(\frac{1}{A_P^2} - 1\right)\right]}{\log\left(\frac{\Omega_S}{\Omega_P}\right)}$$

Proof:

From the squared magnitude spectrum

$$|H(j\Omega)|^2 = \frac{1}{1 + \left(\frac{\Omega}{\Omega_C}\right)^{2N}}$$

At $\Omega = \Omega_P$, $|H(j\Omega_p)| = A_p \Rightarrow |H(j\Omega_P)|^2 = \dfrac{1}{1+\left(\dfrac{\Omega_P}{\Omega_C}\right)^{2N}}$

$$\Rightarrow A_P{}^2 = \frac{1}{1+\left(\frac{\Omega_P}{\Omega_C}\right)^{2N}}$$

$$\Rightarrow \frac{1}{A_P{}^2} = 1+\left(\frac{\Omega_P}{\Omega_C}\right)^{2N}$$

$$\Rightarrow \frac{1}{A_P{}^2} - 1 = \left(\frac{\Omega_P}{\Omega_C}\right)^{2N} \text{ ---------------(1)}$$

At $\Omega = \Omega_S$, $|H(j\Omega_s)| = A_s \Rightarrow |H(j\Omega_S)|^2 = \dfrac{1}{1+\left(\dfrac{\Omega_S}{\Omega_C}\right)^{2N}}$

$$\Rightarrow A_S{}^2 = \frac{1}{1+\left(\frac{\Omega_S}{\Omega_C}\right)^{2N}}$$

$$\Rightarrow \frac{1}{A_S{}^2} = 1+\left(\frac{\Omega_S}{\Omega_C}\right)^{2N}$$

$$\Rightarrow \frac{1}{A_S{}^2} - 1 = \left(\frac{\Omega_S}{\Omega_C}\right)^{2N} \text{ ---------------(2)}$$

Equation 2 by 1, implies

$$\Rightarrow \left(\frac{\Omega_S}{\Omega_C}\right)^{2N} / \left(\frac{\Omega_P}{\Omega_C}\right)^{2N} = \left(\frac{1}{A_S{}^2} - 1\right) / \left(\frac{1}{A_P{}^2} - 1\right)$$

$$\Rightarrow \left(\frac{\Omega_S}{\Omega_P}\right)^{2N} = \left(\frac{1}{A_S{}^2} - 1\right) / \left(\frac{1}{A_P{}^2} - 1\right)$$

$$\Rightarrow 2N \log\left(\frac{\Omega_S}{\Omega_P}\right) = \log\left(\frac{1}{A_S{}^2} - 1\right) \Big/ \left(\frac{1}{A_P{}^2} - 1\right)$$

$$\Rightarrow N = \frac{1}{2}\,\frac{\log\left[\left(\frac{1}{A_S{}^2} - 1\right) \Big/ \left(\frac{1}{A_P{}^2} - 1\right)\right]}{\log\left(\frac{\Omega_S}{\Omega_P}\right)}$$

Note: Choose the order of the filter, such that the number of poles ≥ N.

Ex:

(a) If N = 0.121, then Number of poles = N = 1
(b) If N = 0.981, then Number of poles = N = 1
(c) If N = 1.49, then Number of poles = N = 2
(d) If N = 2.0901, then Number of poles = N = 3

(iv)Cutoff Frequency of ABLPF:

Cutoff frequency of ABLPF can be computed from the formula

$$\Omega_C = \frac{\Omega_s}{\left(\frac{1}{A_s^{\,2}} - 1\right)^{\frac{1}{2N}}}$$

Proof:

At $\Omega = \Omega_s$, $|H(j\Omega s)| = As$

$$\Rightarrow |H(j\Omega_S)|^2 = \frac{1}{1+\left(\frac{\Omega_S}{\Omega_C}\right)^{2N}}$$

$$\Rightarrow A_S^{\,2} = \frac{1}{1+\left(\frac{\Omega_S}{\Omega_C}\right)^{2N}}$$

$$\Rightarrow \frac{1}{A_S^{\,2}} = 1+\left(\frac{\Omega_S}{\Omega_C}\right)^{2N}$$

$$\Rightarrow \frac{1}{A_S^{\,2}} - 1 = \left(\frac{\Omega_S}{\Omega_C}\right)^{2N}$$

$$\Rightarrow \frac{\Omega_S}{\Omega_C} = \left(\frac{1}{A_S^{\,2}} - 1\right)^{\frac{1}{2N}}$$

$$\Rightarrow \Omega_C = \frac{\Omega_S}{\left(\frac{1}{A_S^{\,2}} - 1\right)^{\frac{1}{2N}}}$$

(v)Normalized Transfer Function of ABLPF:

Normalized transfer function of analog filter can be obtained from the formula

$$\text{(a) If N is even, then } H(s_n) = \prod_{k=1}^{N/2}\left(\frac{1}{s_n^2 + b_k s_n + 1}\right)$$

$$\text{(b) If N is odd, then } H(s_n) = \left(\frac{1}{s_n + 1}\right)\prod_{k=1}^{\frac{N-1}{2}}\left(\frac{1}{s_n^2 + b_k s_n + 1}\right)$$

$$\text{where, } b_k = 2\,\text{Sin}\left(\frac{(2k-1)\pi}{2N}\right), k = 1,2,3,4......b_1 = 2\,\text{Sin}\left(\frac{\pi}{2N}\right), b_2 = 2\,\text{Sin}\left(\frac{3\pi}{2N}\right)$$

Examples:

$$\text{(a) First Order Filter} \Rightarrow H(s_n) = \left(\frac{1}{s_n + 1}\right)$$

$$\text{(b) Second order filter} \Rightarrow H(s_n) = \left(\frac{1}{s_n^2 + b_1 s_n + 1}\right)$$

$$(c)\ \text{Third order filter} \Rightarrow H(s_n) = \left(\frac{1}{s_n+1}\right)\left(\frac{1}{s_n^2+b_1s_n+1}\right)$$

$$(d)\ \text{Fourth order filter} \Rightarrow H(s_n) = \left(\frac{1}{s_n^2+b_1s_n+1}\right)\left(\frac{1}{s_n^2+b_2s_n+1}\right)$$

(B)Chebyshev Approximation:

There are two types of Chebyshev approximations.

- Chebyshev Type – I or Chebyshev approximation
- Chebyshev Type – II or Inverse Chebyshev approximation

Analog Chebyshev type – I or Analog Chebyshev Low Pass Flter (ACLPF) is designed by approximating the magnitude of frequency response $|H(i\Omega)|$ is selected such that the magnitude response is equiripple in the pass band and monotonic in the stop band.

Analog Chebyshev type – II or Analog Inverse Chebyshev Low Pass Flter (AICLPF) is designed by approximating the magnitude of frequency response $|H(i\Omega)|$ is selected such that the magnitude response is monotonic in the pass band and equiripple in the stop band.

The approximated squared magnitude frequency response of Chebyshev type – I low pass filter is given by

$$|H(j\Omega)|^2 = \frac{1}{1+\varepsilon\ C_N^2\left(\frac{\Omega}{\Omega_C}\right)}$$

where,

$$\varepsilon = \sqrt{\frac{1}{A_P^2} - 1},$$

$$\text{For small values of N,}\quad C_N(x) = \begin{cases} \text{Cos}[\,N\ \text{Cos}^{-1}(x)\,], & \text{for } |x| \le 1 \\ \text{Cosh}[\,N\ \text{Cosh}^{-1}(x)\,], & \text{for } |x| > 1 \end{cases}$$

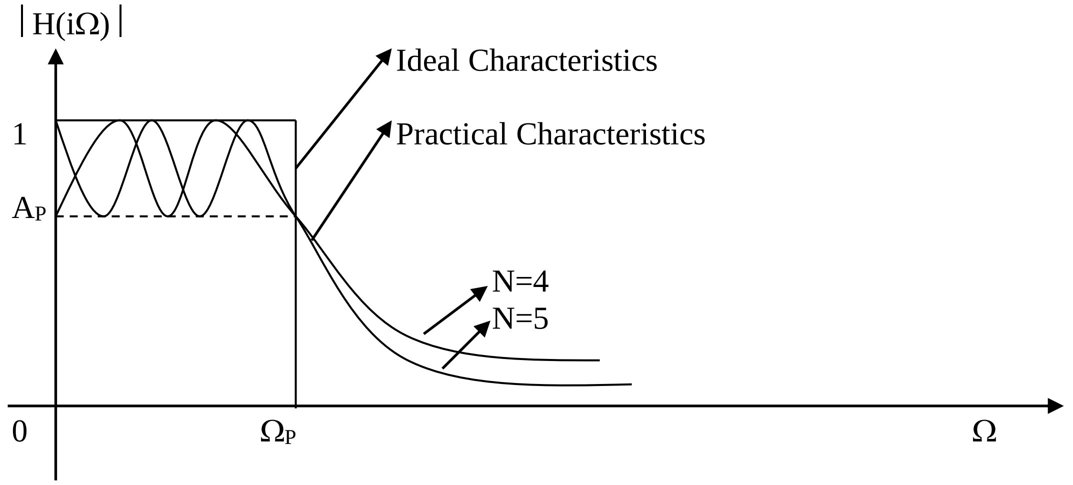
$$\text{For large values of N,}\quad C_N(x) = 2x\, C_{N-1}(x) - C_{N-2}(x),$$

with initial conditions $C_0(x) = 1$ and $C_1(x) = x$.

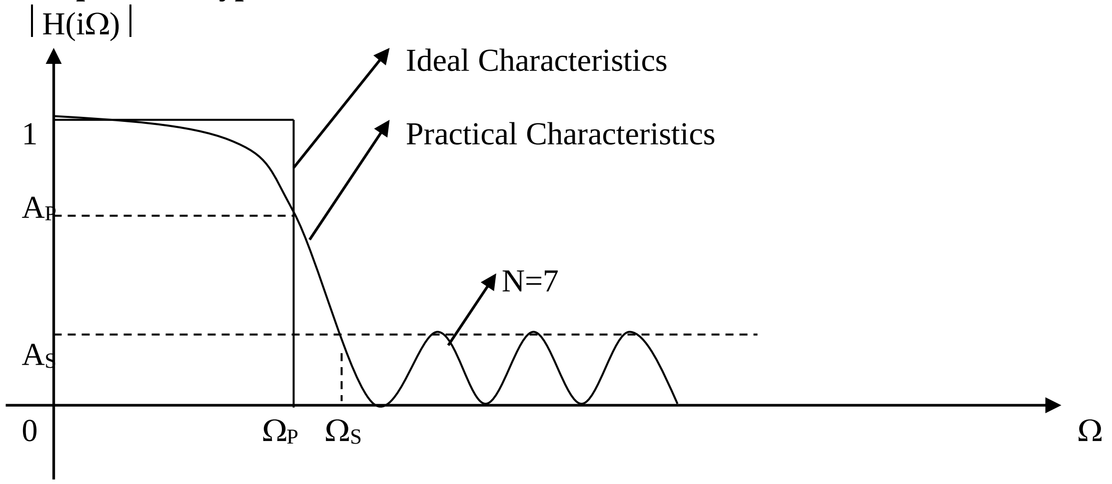
Ω_C	: Cutoff frequency of an analog filter in rad/sec
N	: Order of the filter (Number of poles)
ε	: Attenuation constant
$C_N(\Omega / \Omega_C)$	: Chebyshev polynomial

(i)Magnitude Response of Type – I ACLPF:

Magnitude response $|H(i\Omega)|$ Vs Ω for different N values as shown, where the magnitude is equiripple in the pass band and monotonic in the stop band.



(ii)Magnitude Response of Type – II ACLPF:



(iii)Properties of ACLPF:

- Chebyshev filter is a all pole design.
- All the poles lie on an ellipse in s-plane.
- At the cutoff frequency ($\Omega = \Omega_C = \Omega_P$), the magnitude response $|H(i\Omega)| = A_P$.
- Order (N) of the filter is depends on specifications of the desired filter
- Type – I Magnitude frequency response is equiripple in the pass band and monotonic in the stop band.
- Type – II Magnitude frequency response is monotonic in the pass band and equiripple in the stop band.
- The magnitude response approaches the ideal response as the order of the filter increases.

(iv)Order of ACLPF

Order of the filter is nothing but number of poles of desired filter and it can be computed from the formula.

$$N = \frac{\text{Cosh}^{-1}\left[\sqrt{\left(\frac{1}{A_S^2}-1\right)\Big/\left(\frac{1}{A_P^2}-1\right)}\right]}{\text{Cosh}^{-1}\left(\frac{\Omega_S}{\Omega_P}\right)}$$

Note: Choose the order of the filter, such that the number of poles ≥ N.

Ex:

(a) If N = 0.121, then Number of poles = N = 1
(b) If N = 0.981, then Number of poles = N = 1
(c) If N = 1.49, then Number of poles = N = 2
(d) If N = 2.0901, then Number of poles = N = 3

(v)Cutoff Frequency of ACLPF:

Cutoff frequency of ACLPF can be computed from the formula

$$\Omega_C = \frac{\Omega_s}{\left(\frac{1}{A_s^2}-1\right)^{\frac{1}{2N}}}$$

(v)Normalized Transfer Function of ACLPF:

Normalized transfer function of analog filter can be obtained from the formula

$$\text{(a) If N is even, then } H(s_n) = \prod_{k=1}^{N/2}\left(\frac{b_0}{s_n^2 + b_k s_n + c_k}\right)$$

$$\text{(b) If N is odd, then } H(s) = \left(\frac{b_0}{s_n + c_0}\right)\prod_{k=1}^{\frac{N-1}{2}}\left(\frac{b_0}{s_n^2 + b_k s + c_k}\right)$$

$$\text{where, } b_k = 2\,a\,\text{Sin}\left(\frac{(2k-1)\pi}{2N}\right),$$

$$c_k = a^2 + \text{Cos}^2\left(\frac{(2k-1)\pi}{2N}\right),$$

$$c_0 = a$$

$$a = \frac{1}{2}\left[\left(\sqrt{\frac{1}{\varepsilon^2}+1}+\frac{1}{\varepsilon}\right)^{\frac{1}{N}} - \left(\sqrt{\frac{1}{\varepsilon^2}+1}+\frac{1}{\varepsilon}\right)^{-\frac{1}{N}}\right]$$

$$\varepsilon = \sqrt{\left(\frac{1}{Ap}\right)^2 - 1}$$

$$\text{Calculate } b_0 \text{ through } H(0) = \begin{cases} \text{Ap, If N is Even} \\ \text{1, If N is Odd} \end{cases}$$

Examples:

$$\text{(a) First Order Filter} \Rightarrow H(s_n) = \left(\frac{b_0}{s_n + c_0}\right)$$

$$\text{(b) Second order filter} \Rightarrow H(s_n) = \left(\frac{b_0}{s_n^2 + b_1 s_n + c_1}\right)$$

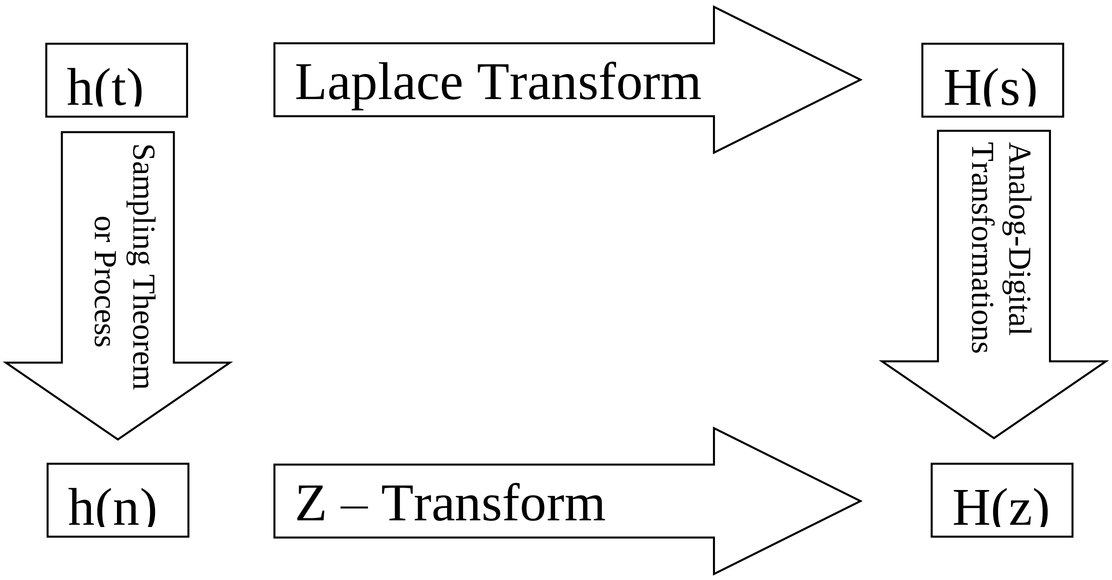
$$\text{(c) Third order filter} \Rightarrow H(s_n) = \left(\frac{b_0}{s_n + c_0}\right)\left(\frac{b_0}{s_n^2 + b_1 s_n + c_1}\right)$$

$$\text{(d) Fourth order filter} \Rightarrow H(s_n) = \left(\frac{b_0}{s_n^2 + b_1 s_n + c_1}\right)\left(\frac{b_0}{s_n^2 + b_2 s_n + c_1}\right)$$

Analog to Digital Transformation Techniques:

The transfer function of digital filter H(z) can be obtained from the transfer function of analog filter H(s) by using Analog to digital transformation techniques. There are two methods, which are used to obtain H(z) from H(s).

- Impulse Invariant Transformation
- Bilinear Transformation



(A)Impulse Invariant Transformation:

Let us consider an analog filter has N poles, which are located at $s = p_1, p_2, p_3, \ldots p_N$, implies,

$$H(s) = \frac{n(s)}{(s-p_1)(s-p_2)(s-p_3)\ldots\ldots\ldots(s-p_N)}$$

Split into partial fractions

$$= \frac{A_1}{s-p_1} + \frac{A_2}{s-p_2} + \frac{A_3}{s-p_3} + \ldots\ldots\ldots + \frac{A_N}{s-p_N}$$

$$= \sum_{k=1}^{N} \frac{A_k}{s-p_k} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Obtain the impulse response of analog filter by using Inverse Laplace Transform

$$L^{-1}[H(s)] = \sum_{k=1}^{N} L^{-1}\left(\frac{A_k}{s-p_k}\right)$$

$$h(t) = \sum_{k=1}^{N} A_k e^{-p_k t}\, u(t)$$

Obtain the impulse response of digital filter by substituting t with nT

$$h(nT) = \sum_{k=1}^{N} A_k e^{p_k nT} u(nT)$$

$$= \sum_{k=1}^{N} A_k \left(e^{p_k T}\right)^n u(nT)$$

Determine the transformation of digital filter H(z) by using z transform

$$ZT\,[h(nT)] = \sum_{k=1}^{N} ZT\,[A_k \left(e^{p_k T}\right)^n u(nT)]$$

$$H(z) = \sum_{k=1}^{N} A_k \left(\frac{z}{z - e^{p_k T}}\right)$$

$$H(z) = \sum_{k=1}^{N} \left(\frac{A_k}{1 - e^{p_k T} z^{-1}}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Compare equations 1 and 2

$$\frac{1}{s-p_k} \to \frac{1}{1 - e^{p_k T} z^{-1}}$$

In Impulse Invariant Transformation, the transfer function of digital filter H(z) can be obtained from the transfer function of analog filter H(s) by using the following conversion formula.

$$\frac{1}{s-p} \to \frac{1}{1 - e^{pT} z^{-1}}$$

where, T is the sampling period

(i)Relation between Analog Frequency (Ω) & Digital Frequency (ω):

We know that the pole of analog filter, which is located at s = p is transformed to pole of digital filter, which is located at $z = e^{pT}$.

$$\Rightarrow z = e^{sT}$$

Substitute $z = r\, e^{j\omega}$ and $s = \sigma + j\Omega$

$$\Rightarrow r\, e^{j\omega} = e^{(\sigma + j\Omega)T}$$

$$\Rightarrow r\, e^{j\omega} = e^{\sigma T} e^{j\Omega T}$$

Compare magnitude and phase

$$\Rightarrow r = e^{\sigma T} \text{ and } \omega = \Omega T \text{ or } \Omega = \frac{\omega}{T}$$

It is the relation between analog frequency and digital frequency.

(ii)Relation between the location of analog and digital filter poles:

We know that the pole of analog filter, which is located at s = p is transformed to pole of digital filter, which is located at $z = e^{pT}$.

$$\Rightarrow z = e^{sT}$$

$$\text{Take } s = \sigma + j\Omega$$

$$\Rightarrow z = e^{(\sigma + j\Omega)T}$$

$$\Rightarrow z = e^{\sigma T} e^{j\Omega T}$$

$$\Rightarrow |z| = e^{\sigma T}$$

Case 1 : $\sigma < 0 \Rightarrow |z| < 1$.

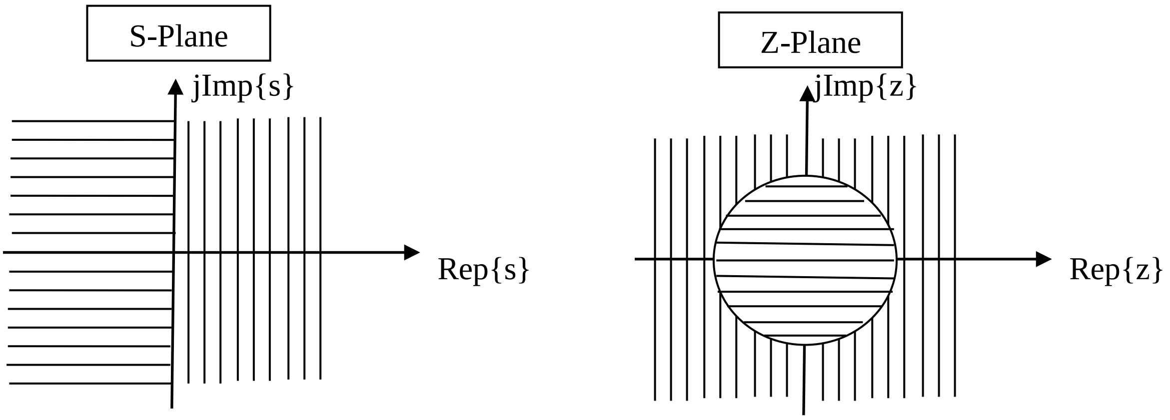
If the pole of analog filter is left sided, then the corresponding pole of digital filter is inside the unit circle. This case belongs to stable system.

Case 2 : $\sigma > 0 \Rightarrow |z| > 1$.

If the pole of analog filter is right sided, then the corresponding pole of digital filter is outside the unit circle. This case belongs to unstable.

Case 3 : $\sigma = 0 \Rightarrow |z| = 1$.

If the pole of analog filter is on the imaginary axes of s-plane, then the corresponding pole of digital filter is on the unit circle of z-plane. This case belongs to marginally stable.



(iii)Frequency Aliasing:

If $s_k = \sigma + j\Omega + j\frac{2\pi}{T}k, k = 0, \pm 1, \pm 2, \ldots\ldots.$ and $z_k = e^{s_k T} = e^{(\sigma + j\Omega + j\frac{2\pi}{T}k)T}$

then

$s_0 = \sigma + j\Omega$ and $z_0 = e^{(\sigma + j\Omega)T} = e^{\sigma T} e^{j\Omega T}$

$s_1 = \sigma + j\Omega + j\frac{2\pi}{T}$ and $z_1 = e^{s_1 T} = e^{(\sigma + j\Omega + j\frac{2\pi}{T})T} = e^{\sigma T} e^{j\Omega T} = z_0$

$s_2 = \sigma + j\Omega + j\frac{4\pi}{T}$ and $z_2 = e^{s_2 T} = e^{(\sigma + j\Omega + j\frac{4\pi}{T})T} = e^{\sigma T} e^{j\Omega T} = z_1 = z_0$

.....

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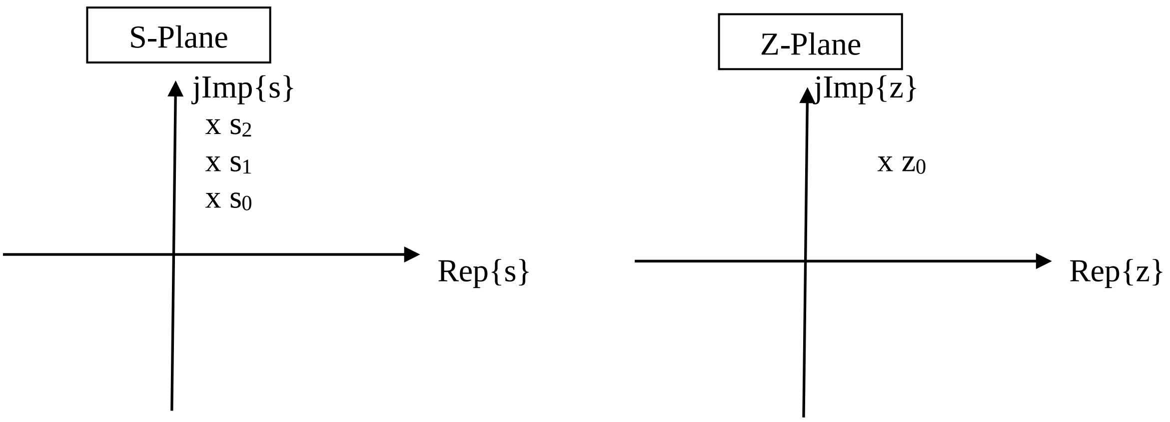
It is very clear from above assumption is that the digital filter pole is unique for the different analog filter poles located at $s_k = \sigma + j\Omega + j\frac{2\pi}{T}k$, where $k = 0, \pm 1, \pm 2, \ldots\ldots$



Impulse invariant transformation is a many-to-one type of conversion or mapping method, because it maps many poles in s-domain into single pole in z-domain. Effect of getting single digital frequency for various analog frequencies is known as frequency aliasing.

(B)Bilinear Transformation:

In bilinear transformation, the transfer function of digital filter H(z) can be obtained from the transfer function of analog filter H(s) by using the formula

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

Proof:

We know that the RC-high pass filter acts as a differentiator

Where the output y(t) is the differentiation of input x(t), i.e $y(t) = \frac{d}{dt}[\,x(t)\,]$

Apply Laplace transform both side to get the transfer function of analog filter H(s)

$$\Rightarrow LT[\,y(t)\,] = LT\left(\frac{d}{dt}[\,x(t)\,]\right)$$

$$\Rightarrow Y(s) = s\,X(s)$$

$$\Rightarrow \frac{Y(s)}{X(s)} = s$$

$$\Rightarrow H(s) = s \text{ - } (1)$$

Convert the differential equation $y(t) = \frac{d}{dt}[\,x(t)\,]$ into difference equation by applying integration over the range two successive discrete samples (n – 1)T and nT.

$$\Rightarrow \int_{(n-1)T}^{nT} y(t)\; dt = \int_{(n-1)T}^{nT} \frac{d}{dt}[\,x(t)\,]\,dt$$

$$\Rightarrow \int_{(n-1)T}^{nT} y(t)\; dt = x(t)\Big|_{(n-1)T}^{nT}$$

Use trapezoidal rule of the numerical integration $\int_{(a}^{b} f(t)\; dt = \frac{b-a}{2}[\,y(b)+y(a)\,]$

$$\Rightarrow \frac{nT\text{ - }(n\text{-}1)T}{2}[\,y(nT)+y((n-1)T)\,] = x(nT) - x((n-1)T)$$

$$\Rightarrow \frac{T}{2}[\,y(nT)+y((n-1)T)\,] = x(nT) - x((n-1)T)$$

Apply Z transform both side to get the transfer function of digital filter H(z)

$$\Rightarrow \frac{T}{2}[\,Y(z)+z^{-1}\,Y(z)\,] = X(z) - z^{-1}\,X(z)$$

$$\Rightarrow \frac{T}{2}[\,1+z^{-1}\,]\,Y(z) = [\,1-z^{-1}\,]\,X(z)$$

$$\Rightarrow \frac{Y(z)}{X(z)} = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$\Rightarrow H(z) = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) \text{– – – – – – – – – – – – – – –} (2)$$

Compare equations (1) and (2) to get the relation between s and z

$$\Rightarrow s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

Bilinear transformation is a one-to-one type of mapping method.

(i)Relation between Analog Frequency (Ω) & Digital Frequency (ω):

We know that

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right),$$

Substitute $s = \sigma + j\Omega$ and $z = r\,e^{j\omega}$

$$\sigma + j\Omega = \frac{2}{T}\left(\frac{1-(r\,e^{j\omega})^{-1}}{1+(r\,e^{j\omega})^{-1}}\right)$$

on the imaginary axis of s-plane σ = 0 and the corresponding z plane is unit circle, where r = 1.

$$\Rightarrow j\Omega = \frac{2}{T}\left(\frac{1-(e^{j\omega})^{-1}}{1+(e^{j\omega})^{-1}}\right)$$

$$\Rightarrow j\Omega = \frac{2}{T}\left(\frac{1-e^{-j\omega}}{1+e^{-j\omega}}\right)$$

$$\Rightarrow j\Omega = \frac{2}{T}\,\frac{e^{-j\frac{\omega}{2}}(e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}})}{e^{-j\frac{\omega}{2}}(e^{j\frac{\omega}{2}} + e^{-j\frac{\omega}{2}})}$$

$$\Rightarrow j\Omega = \frac{2}{T}\left(\frac{e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}}{e^{j\frac{\omega}{2}} + e^{-j\frac{\omega}{2}}}\right)$$

$$\Rightarrow j\Omega = \frac{2}{T}\left(\frac{2j\ \mathrm{Sin}\left(\frac{\omega}{2}\right)}{2\,\mathrm{Cos}\left(\frac{\omega}{2}\right)}\right)$$

$$\Rightarrow \Omega = \frac{2}{T}\left(\frac{\mathrm{Sin}\left(\frac{\omega}{2}\right)}{\mathrm{Cos}\left(\frac{\omega}{2}\right)}\right)$$

$$\Rightarrow \Omega = \frac{2}{T}\,\mathrm{Tan}\left(\frac{\omega}{2}\right) \quad \text{or} \quad \omega = 2\mathrm{Tan}^{-1}\left(\frac{\Omega T}{2}\right)$$

(ii)Relation between the location of analog and digital filter poles:

We know that $$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) \Rightarrow z = \frac{1+\frac{T}{2}s}{1-\frac{T}{2}s}$$

$$\Rightarrow z = \frac{1+\frac{T}{2}(\sigma + j\omega)}{1-\frac{T}{2}(\sigma + j\omega)}$$

$$\Rightarrow z = \frac{1+\frac{T}{2}\sigma + j\frac{T}{2}\omega}{1-\frac{T}{2}\sigma - j\frac{T}{2}\omega}$$

$$\Rightarrow |z| = \frac{\sqrt{\left(1+\frac{T}{2}\sigma\right)^2 + \left(\frac{T}{2}\omega\right)^2}}{\sqrt{\left(1-\frac{T}{2}\sigma\right)^2 + \left(\frac{T}{2}\omega\right)^2}}$$

Case 1 : $\sigma < 0 \Rightarrow |z| < 1$.

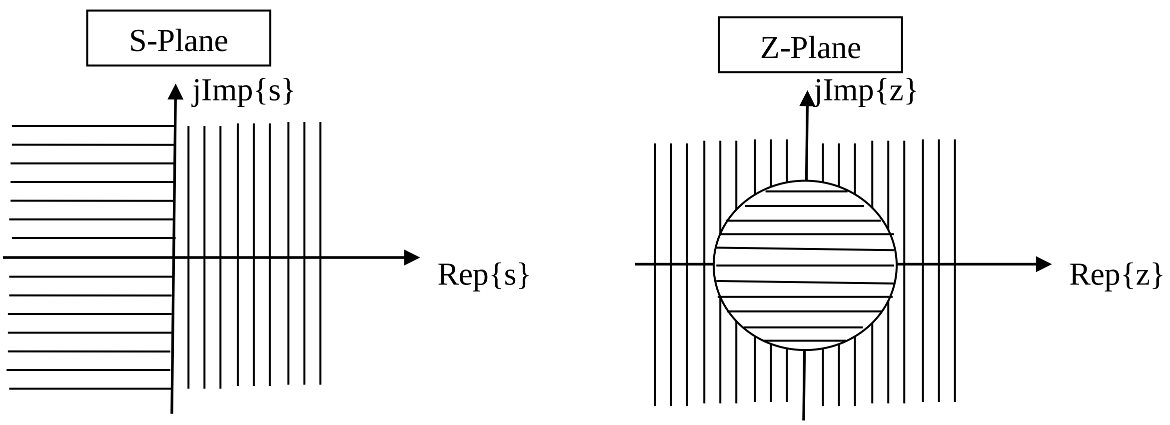
If the pole of analog filter is left sided, then the corresponding pole of digital filter is inside the unit circle. This case belongs to stable system.

Case 2 : $\sigma > 0 \Rightarrow |z| > 1$.

If the pole of analog filter is right sided, then the corresponding pole of digital filter is outside the unit circle. This case belongs to unstable.

Case 3 : $\sigma = 0 \Rightarrow |z| = 1$.

If the pole of analog filter is on the imaginary axes of s-plane, then the corresponding pole of digital filter is on the unit circle of z-plane. This case belongs to marginally stable.

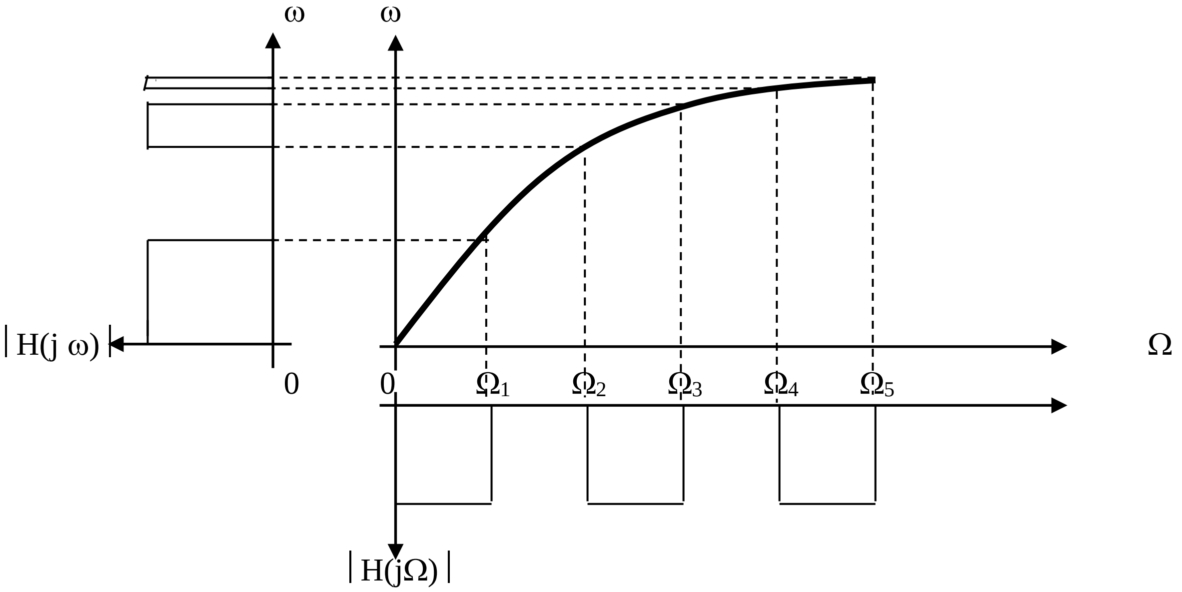


(iii)Frequency Warping:

Relation between analog frequency and digital frequency of bilinear transformation is given by

$$\Omega = \frac{2}{T}\,\text{Tan}\left(\frac{\omega}{2}\right) \quad \text{or} \quad \omega = 2\text{Tan}^{-1}\left(\frac{\Omega T}{2}\right)$$

Draw the graph by taking the digital frequency(ω) on y-axes and analog frequency(Ω) on x-axes.



The relation between analog frequency and digital frequency is non linear. When the s-plane is mapped into the z-plane using bilinear transformation, the non-linear relationship introduces the distortion in frequency axis with disproportionate bandwidth is called frequency warping.

Frequency Transformations:

Transfer function of an analog LPF/HPF/BPF/BSF can be obtained from the normalized transfer function of analog LPF $H(s_n)$ by using Frequency transformation techniques.

(a)For a low pass filter, replace s_n with s / Ω_C.

(b)For a high pass filter, replace s_n with Ω_C / s.

(c)For a band pass filter, replace s_n with $Q(s^2 + \Omega_0^2) / \Omega_0 s$.

(d)For a band stop filter, replace s_n with $\Omega_0 s / Q(s^2 + \Omega_0^2)$.

Where,

Ω_C	: Cutoff frequency of a filter
Ω_0	: Center frequency, $\Omega_0 = \sqrt{\Omega_L \Omega_H}$
Q	: Quality factor, $Q = \Omega_0 / (\Omega_H - \Omega_L)$
Ω_L	: Lower cutoff frequency
Ω_H	: Upper cutoff frequency

Design of Digital LPF/HPF/BPF/BSF through IIR Approximation Methods:

Step 1 :

Take the specifications of a digital low pass filter

ω_P : Pass band digital frequency in rad/sample.

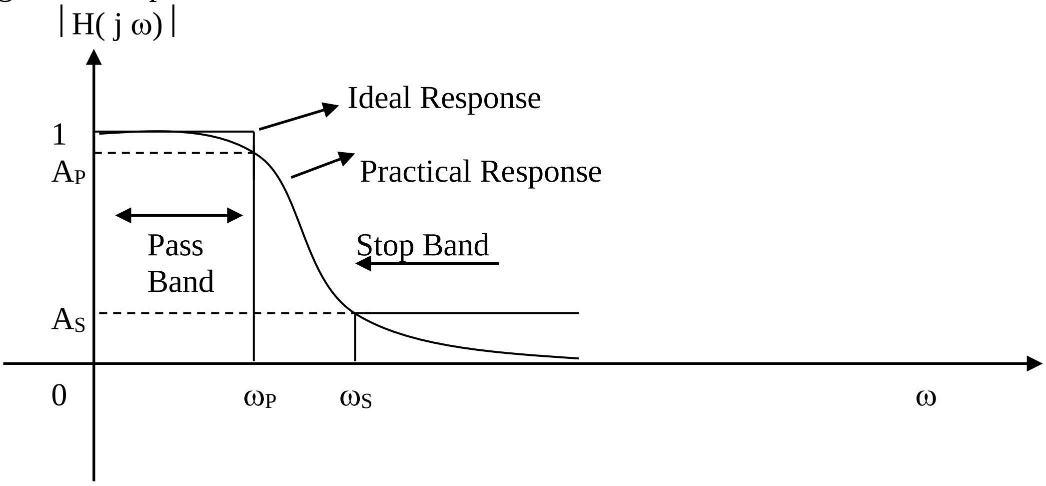
ω_S : Stop band digital frequency in rad/sample.

A_P : Gain at pass band digital frequency ω_P.

A_S : Gain at stop band digital frequency ω_S.

T : Sampling period in sec. (Default value, T = 1 sec)

Magnitude Response of LPF:



Step 2 :

Determine analog frequencies Ω_s and Ω_p in rad/sec by using analog to digital transformation techniques.

(A)Impulse Invariant Transformation

$$\Omega = \frac{\omega}{T} \Rightarrow \Omega_S = \frac{\omega_s}{T} \text{ and } \Omega_p = \frac{\omega_p}{T}$$

(B)Bilinear Transformation

$$\Omega = \frac{2}{T}\text{Tan}\left(\frac{\omega}{2}\right) \Rightarrow \Omega_S = \frac{2}{T}\text{Tan}\left(\frac{\omega_S}{2}\right) \& \Omega_P = \frac{2}{T}\text{Tan}\left(\frac{\omega_P}{2}\right)$$

Step 3:

Decide the order (N) of the filter by using analog filter approximation methods

(A)Analog Butterworth Approximation

$$N = \frac{1}{2}\frac{\log\left[\left(\frac{1}{A_S{}^2} - 1\right) \Big/ \left(\frac{1}{A_P{}^2} - 1\right)\right]}{\log\left(\frac{\Omega_S}{\Omega_P}\right)}$$

(B)Analog Chebyshev Approximation

$$N = \frac{\cosh^{-1}\left[\sqrt{\left(\frac{1}{A_S^{\,2}}-1\right)\Big/\left(\frac{1}{A_P^{\,2}}-1\right)}\right]}{\cosh^{-1}\left(\frac{\Omega_S}{\Omega_P}\right)}$$

Step 4:

Determine the normalized transfer function of an analog filter by using analog filter approximation methods.

(A)Analog Butterworth Approximation

$$\text{(a) If N is even, then } H(s_n) = \prod_{k=1}^{N/2}\left(\frac{1}{s_n^2 + b_k s_n + 1}\right)$$

$$\text{(b) If N is odd, then } H(s_n) = \left(\frac{1}{s_n + 1}\right)\prod_{k=1}^{\frac{N-1}{2}}\left(\frac{1}{s_n^2 + b_k s_n + 1}\right)$$

$$\text{where, } b_k = 2\,\text{Sin}\left(\frac{(2k-1)\pi}{2N}\right), k = 1,2,3,4......b_1 = 2\,\text{Sin}\left(\frac{\pi}{2N}\right), b_2 = 2\,\text{Sin}\left(\frac{3\pi}{2N}\right)$$

(B)Analog Chebyshev Approximation

$$\text{(a) If N is even, then } H(s_n) = \prod_{k=1}^{N/2}\left(\frac{b_0}{s_n^2 + b_k s_n + c_k}\right)$$

$$\text{(b) If N is odd, then } H(s) = \left(\frac{b_0}{s_n + c_0}\right)\prod_{k=1}^{\frac{N-1}{2}}\left(\frac{b_0}{s_n^2 + b_k s + c_k}\right)$$

$$\text{where, } b_k = 2a\,\text{Sin}\left(\frac{(2k-1)\pi}{2N}\right),$$

$$c_k = a^2 + \text{Cos}^2\left(\frac{(2k-1)\pi}{2N}\right),$$

$$c_0 = a$$

$$a = \frac{1}{2}\left[\left(\sqrt{\frac{1}{\varepsilon^2}+1}+\frac{1}{\varepsilon}\right)^{\frac{1}{N}} - \left(\sqrt{\frac{1}{\varepsilon^2}+1}+\frac{1}{\varepsilon}\right)^{-\frac{1}{N}}\right]$$

$$\varepsilon = \sqrt{\left(\frac{1}{Ap}\right)^2 - 1}$$

$$\text{Calculate } b_0 \text{ through } H(0) = \begin{cases} \text{Ap, If N is Even} \\ \text{1, If N is Odd} \end{cases}$$

Step 5:

Determine the transfer function of an analog LPF/HPF/BPF/BSF through normalized transfer function of analog LPF $H(s_n)$ by using frequency transformation techniques.

(a)For a low pass filter, replace s_n with s / Ω_C.

(b)For a high pass filter, replace s_n with Ω_C / s.

(c)For a band pass filter, replace s_n with $Q(s^2 + \Omega_0^2) / \Omega_0 s$.

(d)For a band stop filter, replace s_n with $\Omega_0 s / Q(s^2 + \Omega_0^2)$.

Where,

Ω_C	: Cutoff frequency of a filter
Ω_0	: Center frequency, $\Omega_0 = \sqrt{\Omega_L \Omega_H}$
Q	: Quality factor, $Q = \Omega_0 / (\Omega_H - \Omega_L)$
Ω_L	: Lower cutoff frequency
Ω_H	: Upper cutoff frequency

$$\Omega_C = \frac{\Omega_S}{\left(\frac{1}{A_S^2} - 1\right)^{\frac{1}{2N}}}$$

Step 6:

Convert the transfer function of analog filter H(s) into the transfer function of digital filter H(z) by using analog to digital transformation techniques.

(A)Impulse Invariant Transformation

$$\frac{1}{s-p} \rightarrow \frac{1}{1-e^{pT}z^{-1}}$$

(B)Bilinear Transformation

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

Step 7:

Finally design the digital filter by a suitable structure (DF-I / DF-II / Cascade Form / Parallel Form)

DESCRIPTIVE QUESTIONS:

1. Design a digital low pass filter by using IIR Butterworth approximation and Bilinear transformation method by taking the sampling period T=0.1 sec to satisfy the following specifications

   $0.6 \le |H(jw)| \le 1.0; 0 \le w \le 0.35\pi,$

   $|H(jw)| \le 0.1; 0.7\pi \le w \le \pi$

2. Design a digital low pass filter by using IIR Butterworth approximation and Bilinear transformation method by taking the sampling period T=0.5 sec to satisfy the following specifications

   $0.707 \le |H(jw)| \le 1.0; 0 \le w \le 0.45\pi,$

   $|H(jw)| \le 0.2; 0.65\pi \le w \le \pi$

3. Design a digital low pass filter by using IIR Butterworth approximation and impulse invariant transformation method by taking the sampling period T=1 sec to satisfy the following specifications

   $0.707 \le |H(jw)| \le 1.0; 0 \le w \le 0.3\pi,$

   $|H(jw)| \le 0.2; 0.75\pi \le w \le \pi$

4. Design a digital low pass filter by using IIR Butterworth approximation and impulse invariant transformation method by taking the sampling period T=1 sec to satisfy the following specifications

   $0.9 \le |H(jw)| \le 1.0; 0 \le w \le 0.35\pi,$

   $|H(jw)| \le 0.275; 0.7\pi \le w \le \pi$

5. Design a digital low pass filter by using IIR Chebyshev approximation and impulse invariant transformation method by taking the sampling period T=1 sec to satisfy the following specifications

   $0.9 \le |H(jw)| \le 1.0; 0 \le w \le 0.25\pi,$

   $|H(jw)| \le 0.24; 0.5\pi \le w \le \pi$

6. Design a digital low pass filter by using IIR Chebyshev approximation and Bilinear transformation method by taking the sampling period T=1 sec to satisfy the following specifications

   $0.8 \le |H(jw)| \le 1.0; 0 \le w \le 0.2\pi,$

   $|H(jw)| \le 0.2; 0.32\pi \le w \le \pi$

7. Design a digital filter by converting the transfer function of analog filter H(s) into digital filter H(z) by using impulse invariant transformation method with a sampling period (a)T=1 sec (b)T=0.1 sec. Given $H(s) = \frac{2}{s^2 + 3s + 2}$

8. Design a digital filter by converting the transfer function of analog filter H(s) into digital filter H(z) by using impulse invariant transformation method. Given $(a) H(s) = \frac{s + 0.1}{s^2 + 0.2s + 9.01}$ $(b) H(s) = \frac{3}{s^2 + 0.2s + 9.01}$

9. Design a digital filter by converting the transfer function of analog filter H(s) into digital filter H(z) by using bilinear transformation method with a sampling period (a)T=1 sec (b)T=0.1 sec. Given $H(s) = \frac{2}{s^2 + 3s + 2}$

10. Design a digital filter by converting the transfer function of analog filter H(s) into digital filter H(z) by using bilinear transformation method. Given

$$H(s) = \frac{s^3}{(s+1)(s^2 + s + 1)}$$

QUIZ QUESTIONS:

1.	Match the following (a)Butterworth Filter (b)Chebyshev Filter (i)Poles lie on a circle in s-plane (ii) Poles lie on an ellipse in s-plane. (iii) At the cutoff frequency $\Omega = \Omega_C$, the magnitude response $\lvert H(i\Omega)\rvert = 0.707$. (iv) At the cutoff frequency $\Omega = \Omega_C$, the magnitude response $\lvert H(i\Omega)\rvert = A_P$	a-i&iii b-ii&iv
2.	In Butterworth Filter, the magnitude response approaches the ideal response as the order of the filter_____	Increases
3.	In which filter, the large value of N, the transition from pass band to stop band becomes more sharp and approaches ideal characteristics	Chebyshev
4.	In which filter, large number of parameters has to be calculated to determine the transfer function	Chebyshev

5.	Which filter is called all pole design (A)Butterworth Filter (B)Chebyshev Filter	A & B
6.	Match the following (a)Impulse Invariant Transformation (b)Bilinear Transformation (i)It is many-to-one mapping (ii) It is one-to-one mapping. (iii)Relation between analog and digital frequency is linear (iv) Relation between analog and digital frequency is nonlinear (v)Aliasing problem (vi)Frequency warping problem	a-i,iii,v b-ii,iv,vi
7.	Analog frequency Ω and digital frequency ω are measured in___ & ____ (A)radians, radians (B)Hz, Hz (C)rad/sec, rad/sample (D) rad/sample,rad/sec	C
8.	Which of the following statement is true for impulse invariant transformation (a)It is a one to one mapping. (b)Relation between analog and digital frequency is $\omega=\Omega T$ (c)Transform $\Rightarrow \frac{1}{s-a} \rightarrow \frac{1}{1-e^{aT}z^{-1}}$ (d)Suffering from frequency warping (A) a, b (B) b, c (C) c, d (D) a,b,c	B
9.	Which of the following statement is true for bilinear transformation (a)It is one to one mapping. (b)Relation between analog and digital frequency is $\omega=2\ \text{Tan}^{-1}(\Omega T/2)$ (c)Transform $\Rightarrow s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$ (d)suffering from frequency warping. (A) a, b, d (B) a, b, c (C) b, c, d (D) a,b,c,d	D
10.	Which of the following conversion is used to obtain desired HPF transfer function from known LPF transfer function of a IIR filter (A) $s\rightarrow s/\Omega_C$ (B) $s\rightarrow \Omega_C/s$ (C) $s\rightarrow s\Omega_C$ (D) None	B